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UNIVERSITY EXAMINATIONS 2023/2024

THIRD YEAR, SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE IN STATISTICS
AND
FOURTH YEAR, SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE (PURE) AND BACHELOR OF SCIENCE IN MATHEMATICS

SMS 3358: MEASURE AND INTEGRATION

DATE: APRIL 2024 **TIME: 2 HOURS**

INSTRUCTIONS: Answer Question ONE and any other TWO questions.

QUESTION ONE (30 MARKS)

a) Explain the following terms;

i. A sigma-algebra (3 Marks)

ii. Measurable set (1 Mark)

iii. Measurable function (2 Marks)

b) Prove that a sigma-algebra is closed with respect to taking countable intersections. (5 Marks)

c) Prove that the function $f(x) = C, \forall x \epsilon X$ and C is a constant, is measurable. (5 Marks)

d) Let $f: X \to \mathbb{R}$ and let A be a non-measurable subset of $\mathbb{R}$. Define $f(x) = \begin{cases} 1, & x\epsilon A \\ -1, & x\epsilon A^C \end{cases}$

Prove that f^2 and $|f|$ are measurable but f is not measurable $\mathbb{R}$. (6 Marks)

e) State, without the monotone convergence theorem. (3 Marks)

f) Given that subsets A and B of $\mathbb{R}$ are lebesgue measurable, prove that $A \cap B$ amd $A \cup B$ of $\mathbb{R}$ are lebesgue measurable. (5 Marks)

QUESTION TWO (20 MARKS)

a) Let $f: X \to Y$ be a function. If Y is a stigma algebra of subsets of Y, prove that the class $\{f^{-1}(E): E \epsilon Y\}$ is a sigma algebra of subsets of X. (10 Marks)

b) Let $f: X \to \mathbb{R}_e$ be measurable. Prove that $\{x \epsilon X: f(x) = \propto, \propto \epsilon \mathbb{R}_e\}$ is measurable.

(10 Marks)

QUESTION THREE (20 MARKS)

a) i. Define a simple function $f: X \to \mathbb{R}$. (2 Marks)

ii. Define an integrable function f over a measurable space. $(X, \mathfrak{X}\mu)$. (2 Marks)

b) given that f and g are integrable functions, prove that $\forall \propto \epsilon \mathbb{R}$;

i. $\propto f$ and $f + g$ are also integrable. (6 Marks)

ii. $\int \propto f d\mu = \propto \int f d\mu$ (3 Marks)

iii. $\int (f + g)\, d\mu = \int f d\mu + \int g d\mu$ (7 Marks)

QUESTION FOUR (20 MARKS)

a) Given that $(f_n)_1^\infty$ is a sequence of $\mathfrak{X}$ measurable functions $f_n: X \to \mathbb{R}$, prove that'

i. $f_1 \cup f_2 \cup \dots . \cup f_n$ and

ii. $f_1 \cap f_2 \cap \dots . \cap f_n$ are $\mathfrak{X}$-measurable. (8 Marks)

b) Let $f: X \to \mathbb{R}_e$ be $\mathfrak{X}$-measurable function. prove that the truncation function

$$f_A(x) = \begin{cases} f(x), & |f(x) < A| \\ A, & f(x) > A \\ -A, & f(x) < -A \end{cases}$$

is $\mathfrak{X}$-measurable. (8 Marks)

c) Let $(f_n)_1^\infty$ be a sequence of $\mathfrak{X}$ $-$measurable functions $f_n: X \to \mathbb{R}_e$ show that the set of points over which (f_n) converges point wise is $\mathfrak{X} -$ measurable. (4 Marks)

QUESTION FIVE (20 MARKS)

a) Define the terms:

   i. Complete measure (2 Marks)

   ii. Simple function (2 Marks)

b) State without proof the Fatou's Lemma. (2 Marks)

c) Explain the meaning of a proposition holds almost everywhere. (2 Marks)

d) Given that $X = \mathbb{R}$, and M is the set of all measurable subsets of $\mathbb{R}$, prove that the space $(\mathbb{R}, M, \mu)$ is complete. (8 Marks)

e) Prove that every countable set has a measure zero. (4 Marks)