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University Examinations 2023/2024

THIRD YEAR SECOND SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF MATHEMATICS AND COMPUTER SCIENCE AND BACHELOR OF SCIENCE IN MATHEMATICS

SMA 3350: GROUP THEORY I

DATE: APRIL 2024 **TIME: 2 HOURS**

INSTRUCTIONS: *Answer question **one** and any other **two** questions*

QUESTION ONE (30 MARKS)

a) Define the following terms

i. A normal subgroup (3 marks)

ii. Inverse element in a group (1 mark)

iii. Cyclic group (1 mark)

iv. Left coset of a subgroup H of a group G (2 marks)

b) Prove that the system $(\mathbb{Z}_5, +_5)$ is an Abelian group (6 marks)

c) Given that $(H_1,*)$ *and* $(H_2,*)$ are subgroup of the group $(G,*)$, prove that $(H_1 \cap H_2,*)$ is also a subgroup of $(G,*)$ (5 marks)

d) Express the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$$

As a product of transposition (3 marks)

e) i. Define the term homomorphism of groups (2 marks)

ii. Determine whether the following mappings are homomorphism's from a group $(G,*) to (G,*)$

i. Define $\phi : G \to G\, by$ $\phi(x) = 3x + 2$ (3 marks)

ii. Define $\phi : G \to G\, by$ $\phi(x) = 2x^2 + x + 4$ (4 marks)

QUESTION TWO (20 MARKS)

a) Given that $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$

Determine the orbits of f (4 marks)

b) Determine whether the permutation $g = (1\ 6\ 8).(3 4).(2 5 7)$ is even or odd (6 marks)

c) Given that G is a finite group whose order is a prime number P, Show that G is cyclic (5 marks)

d) Prove that a cyclic group is abelian (5 marks)

QUESTION THREE (20 MARK)

a) Let $H = \{1, (1\ 3\}$ be a subgroup of S_3. Find all the left cosets of H in S_3, hence determine the index of H in S_3 (5 marks)

b) Let (G*) be a group. Prove that the identity element in G is unique (4 marks)

c) Given that $H \leq G$, prove that the order of an element $a \in H$ in the same as the order of the element $a \in G$ (4 marks)

d) Prove that the order of each subgroup of a finite group divides the order of the group (7 marks)

QUESTION FOUR (20 MARKS)

a) Define the terms

i. Conjugacy (2 marks)

ii. Equivalent relation (3 marks)

iii. Centre of a group (2 marks)

b) Prove that the relation of conjugacy in a group $(G,*)$ is an equivalent relation (8 marks)

c) Prove that the centre of a group is always a normal subgroup of the group (5 marks)

QUESTION FIVE (20 MARKS)

a) Given that if is a normal subgroup of G, prove that the set of all cosets of H in G, G/H is a group of G (10 marks)'

b) Find all the cosets of $(\mathbb{Z}_3, +_3) in\ \mathbb{Z} and$ show that $\mathbb{Z}_3$ decomposes $\mathbb{Z}$ into disjoint (5 marks)

c) Prove that there is a one to one correspondence between the elements of a subgroup $(H\ *)$of a group (g,*) and those of any coset of H