MURANG'A UNIVERSITY OF TECHNOLOGY

SCHOOL OF PURE, APPLIED AND HEALTH SCIENCES

DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE

UNIVERSITY POSTGRADUATE EXAMINATION

2024/2025 ACADEMIC YEAR

FIRST YEAR **FIRST** SEMESTER EXAMINATION FOR DOCTOR OF PHILOSOPHY IN STATISTICS

AMS 802 – OPTIMAL DESIGN OF EXPERIMENTS

DURATION: 3 HOURS

INSTRUCTIONS TO CANDIDATES:

1. Answer any FOUR questions.
2. Mobile phones are not allowed in the examination room.
3. You are not allowed to write on this examination question paper.

QUESTION ONE (25 MARKS)

(a) On the regression range:

$$\chi = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}4\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \right\},$$

Show that the unique optimal design for θ_1 in Ξ is $\xi\begin{pmatrix}4\\1\end{pmatrix} = \frac{2}{3}, \xi\begin{pmatrix}4\\2\end{pmatrix} = \frac{1}{3}$ (4 marks)

(b) In the line fit model over $\mathcal{T} = [-1; 0]$, show that the unique optimal design for $\theta_1 + \theta_2$ in Ξ is $\xi\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{3}, \xi\begin{pmatrix}1\\0\end{pmatrix} = \frac{2}{3}$ (4 marks)

(c) In the parabola fit model over $\mathcal{T} = [-1; 0]$determine designs that are optimal for $\boldsymbol{c}'\boldsymbol{\theta}$ in Ξ with:

i. $\boldsymbol{c} = (1,1,1)'$ (2 marks)

ii. $\boldsymbol{c} = (1,0,-1)'$ (2 marks)

iii. $\boldsymbol{c} = (1,0,-2)'$ (2 marks)

(d) Consider a trigonometric fit model of degree 2:

$$Y_{ij} = \alpha + \beta_1 \cos(t_i) + \gamma_1 \sin(t_i) + \beta_2 \cos(2t_i) + \gamma_2 \sin(2t_i) + E_{ij}$$

Using $t = \left(0, {}^{\pi}/_{3}, {}^{2\pi}/_{3}, \pi, {}^{4\pi}/_{3}, {}^{5\pi}/_{3}, 2\pi\right)$

i. Compute the moment matrix of the design. (6 marks)

ii. Determine the eigenvalues of the matrix in i) above. (5 marks)

QUESTION TWO (25 MARKS)

(a) Explain the linear parameter subsystem. Verify this by considering the full parameter system $(\theta_1, \theta_2, \theta_3, \theta_4)'$ and the parameter subsystem $\left(\theta_1, \theta_2, {}^{1}/_{2}\,(\theta_3 + \theta_4)\right)'$ (5 marks)

(b) In the model with regression function $f(t) = \begin{pmatrix}t\\1-t\end{pmatrix}$ Over $\mathcal{T} = [0; 1]$ show that the unique optimal design for $\theta_1$in Ξ is $\xi\begin{pmatrix}1\\0\end{pmatrix} = 1$ (5 marks)

(c) For the compact convex set;

$$\boldsymbol{M} = \left\{\begin{bmatrix}1 & 0 & \alpha\\0 & \alpha & 0\\\alpha & 0 & \alpha\end{bmatrix} : \alpha \in (0,1)\right\}$$

and the matrix:

$$\boldsymbol{K} = \begin{bmatrix}0 & 0\\1 & 1\\0 & 1\end{bmatrix}$$

obtain C_K (6 marks)

(d) Consider the two matrices A and B both similar to $\boldsymbol{M}$ in (c) above where $\alpha = 0.4$ for A and $\alpha = 0.2$ for B. Obtain C_A and C_B. Hence find $\phi_0(C_A), \phi_0(C_B), \phi_{-1}(C_A)$ and $\phi_{-1}(C_B)$, then make necessary conclusions. (9 marks)

QUESTION THREE (25 MARKS)

(a) Show that for every positive homogeneous function $\phi: NND(s) \rightarrow \mathbb{R}$, the following statements are equivalent.

i. ϕ is superadditive

ii. ϕ is concave (10 marks)

(b) State and prove the Gauss - Markov theorem (8 marks)

(c) Describe the classical optimality criteria. Hence compute the optimal valves for the first order 2^3 design. (7 marks)

QUESTION FOUR (25 MARKS)

The octagon design below is used in an experiment that has the objective of filling a second order response model:

x_1	x_2	y
1.0	0.0	13
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	8
0.0	1.0	10
$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	9
-1.0	0.0	9
$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	10
0.0	-1.0	13
$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$	11
0	0	9
0	0	7

a) Show that the design is rotatable. (4 marks)

b) Fit the second order response model given that

$$(X'X)^{-1} = \frac{1}{8}\begin{bmatrix} x_0 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ 4 & 0 & 0 & -4 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 7 & 0 & 3 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 4 & 0 & 0 & 3 & 0 & 7 \end{bmatrix}$$

(8 marks)

c) Perform canonical analysis. What type of surface has been found? (7 marks)

d) What operating conditions on x_1 and x_2 lead to the stationary point? (6 marks)

QUESTION FIVE (25 MARKS)

A gasoline blending experiment involved three mixture components. There are no constraints on the mixture proportions and the following 10 run design is used:

x_1	x_2	x_3	y $(mpg$
1	0	0	24.5,25.1
0	1	0	24.8,23.9
0	0	1	22.7,23.6
$1/2$	$1/2$	0	25.1
$1/2$	0	$1/2$	24.3
0	$1/2$	$1/2$	23.5
$1/3$	$1/3$	$1/3$	24.8,24.1
$2/3$	$1/6$	$1/6$	24.2
$1/6$	$2/3$	$1/6$	23.9
$1/6$	$1/6$	$2/3$	23.7

a) Determine the type of design the experimenters used (2 marks)
b) Fit a first order mixture model to the data (13 marks)
c) Is the model fitted in (B) adequate? (10 marks)